Problem 1. *Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.*

(1) *Suppose* $a_n > M$ *for* $n \gg 1$ *and* $\lim a_n = L$. *Then,* $L > M$.
(2) *Suppose* $\lim a_n^2 = L$. *Then,* $\lim a_n = \sqrt{L}$.
(3) *Suppose* $\{a_n b_n\}$ *and* $\{a_n\}$ *converge. Then,* b_n *also converges.*
(4) *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *with* $b_n \neq 0$ *are bounded. Then,* a_n/b_n *is also bounded.*
(5) *Suppose a non-empty set* S *has its supremum. Then, the set* $S^2 = \{s^2 : s \in S\}$ *has its supremum and* $(\sup S)^2 = \sup S^2$.
(6) *A sequence of open intervals* $I_n = (a_n, b_n)$ *satisfies* $I_{n+1} \subset I_n$ *and* $\lim |b_n - a_n| = 0$. *Then, there exists a number* L *such that* $\lim a_n = \lim b_n = L$ *and* $L \in I_n$.
(7) *If* $\lim a_n = M$, *then* $\lim |a_n| = |M|$.
(8*) *A sequence* $\{n^2 a_n\}$ *converges. Then, the series* $\sum a_n$ *converges.*
(9) *Let* a_n *and* b_n *be Cauchy sequences. Then,* $a_n b_n$ *is also a Cauchy sequence.*
(10) *Let* $a_n > 0$ *be a Cauchy sequence. Then,* $\frac{1}{a_n}$ *is also a Cauchy sequence.*
(11) *Let* $f(x)$ *be defined for* $x \approx x_0$ *and* $\lim_{x \to x_0} f(x) = L$. *Then,* $L = f(x_0)$.
(12) *Let* $f(x)$ *be a bounded function defined for* $x \approx 0$. *Then,* $xf(x)$ *is continuous at* 0.
(13) *Suppose that* $f(x)$ *has both of the right- and left- hand limits. Then,* $f(x)$ *has the limit at* x_0.
(14) *Let* $f(x)$ *be an increasing function defined on an interval* $[0, 1]$. *Then,* $f(x)$ *is left-continuous at* 1.
(15*) *Let* $f(x)$ *be an increasing function defined on an interval* $[0, 1]$. *Then,* $f(x)$ *has the left-hand limit at* 1.

Proof for (1). **False.** Let $a_n = \frac{1}{n}$ and $M = 0$. Then, $a_n > M$ and $L = \lim a_n = 0$. Thus, $M = L$, not $L > M$. □

Proof for (2). **False.** Let $a_n = (-1)^n$. Then, $a_n^2 = 1$ and thus $\lim a_n^2 = 1$. However, $a_{2m} = 1$ and $a_{2m+1} = -1$. Therefore, there are subsequences of a_n converging to different limits. Thus, a_n can not converge by the subsequence theorem and the uniqueness of the limit. □

Proof for (3). **False.** Let $a_n = 0$ and $b_n = n$. Then, $a_n b_n = 0$ and $a_n = 0$ are constant sequence and therefore converge. However, b_n diverges. □

Proof for (4). **False.** Let $a_n = 1$ and $b_n = \frac{1}{n}$. Then, $0 \leq a_n, b_n \leq 1$. However, $a_n/b_n = n$ is unbounded. □

Proof for (5). **False.** Let $S = (-\infty, 0]$. Then, $\sup S = 0$. However, $S^2 = (-\infty, +\infty)$ and thus $\sup S^2$ does not exist. □

Proof for (6). **False.** Let $I_n = (0, \frac{1}{n})$. Then, $I_{n+1} \subset I_n$, $\lim |b_n - a_n| = \lim \frac{1}{n} = 0$, and $\lim a_n = \lim b_n = 0$. However, $0 \notin I_n$. □

Proof for (7). **True.** If $M = 0$, then given $\epsilon > 0$ we have $|a_n| = |a_n - 0| < \epsilon$ for $n \gg 1$. Thus, $\big||a_n| - |0|\big| = |a_n| < \epsilon$ for $n \gg 1$.

If $M > 0$, then given $\epsilon > 0$ we have $a_n \geq 0$, $|a_n - M| < \epsilon$ for $n \gg 1$. Thus, $\big||a_n| - |M|\big| = |a_n - M| < \epsilon$ for $n \gg 1$.

If $M < 0$, then given $\epsilon > 0$ we have $a_n \leq 0$, $|a_n - M| < \epsilon$ for $n \gg 1$. Thus, $\big||a_n| - |M|\big| = | - a_n + M| < \epsilon$ for $n \gg 1$. □

Proof for (8). **True.** By the test for divergence, we have $\lim n^2 a_n = 0$. Therefore, we have $|n^2 a_n| \leq 1$ for $n \gg 1$, namely $|a_n| \leq \frac{1}{n^2}$ for $n \geq N$ where N is a large constant.

Since $\sum_N^\infty \frac{1}{n^2}$ converges by the proof of Example 7.5A in page 104, the comparison theorem $\sum_N^\infty |a_n|$ converges. Hence, the tail-convergence theorem $|a_n|$ converges. Therefore, a_n is absolutely convergent. □

Proof for (9). **True.** Since a_n, b_n are Cauchy sequences, they are convergent. Hence, $a_n b_n$ is also convergent to its limit L by the multiplication theorem. Therefore, given $\epsilon > 0$ we have $|a_n b_n - L| < \epsilon/2$ for $n \geq N$. Thus, $|a_n b_n - a_m b_m| < \epsilon$ for $n, m \geq N$. □

Proof for (10). **False.** Let $a_n = 1/n$. Then, $1/a_n = n$ diverges. So, it is not a Cauchy sequence, since every Cauchy sequence must converge. □

Proof for (11). **False.** Let $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. Then, $\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$. □

Proof for (12). **True.** Since $f(x)$ is bounded for $x \approx 0$, there exists some numbers $M, \delta > 0$ such that $|f(x)| \leq M$ holds for $x \in (-\delta, \delta)$. Therefore, given $\epsilon > 0$ we have

$$|xf(x) - 0f(0)| = |x||f(x)| \leq M|x| < \epsilon,$$

for $x \in (-\epsilon/M, \epsilon/M) \cap (-\delta, \delta)$. Namely, $xf(x)$ is continuous at 0. □

Proof for (13). **False.** Let $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x > 0$. Then, $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$. Assume that $f(x)$ has the limit L. Then, there exists $\delta > 0$ such that $|f(x) - L| < 1$ holds for $x \in (-\delta, \delta) \setminus \{0\}$. So, $|1 - L| = |f(\delta/2) - L| < 1$ and $|1 + L| = |f(\delta/2) - L| < 1$. Thus, we have a contradiction $2 \leq |1 - L| + |1 + L| < 2$. □

Proof for (14). **False.** Let $f(x) = x$ for $x \in [0, 1)$ and $f(1) = 2$. Then, $f(x)$ is increasing. However, $|f(x) - 2| = 2 - x \geq 1$ for $x \in [0, 1)$, namely $f(x)$ is not left-continuous at 1. □

Proof for (15). **True.** Let $I = [0, 1)$. Then, $f(1)$ is an upper bound for $f(I)$, since $f(x) \leq f(1)$ for $x \in I$. Thus, by the completeness property of sets, $f(I)$ has the supremum $\bar{m}$.

We now claim that $\bar{m}$ is the left-hand limit of $f(x)$ at 1. For every $n \in \mathbb{N}$, there exists a number $x_n \in I$ such that $\bar{m} - \frac{1}{n} < f(x_n) \leq \bar{m}$. So, given $\epsilon > 0$ we choose a number $N > 1/\epsilon$. Since $f(x)$ is increasing

$$\bar{m} - \epsilon < \bar{m} - \frac{1}{N} < f(x_N) \leq f(x) \leq \sup f(I) = \bar{m},$$

holds for $x \in (x_N, 1) \subset I$. Thus, $|\bar{m} - f(x)| < \epsilon$ for $x \in (1 - x_N, 1)$, namely $\bar{m}$ is the left-hand limit.

□

Problem 2. *Determine whether the following sequences are convergent or divergent. If convergent, find the limit and explain why it is the limit. If divergent, explain why the sequence is not convergent.*

$$(1)\, a_n = \frac{(-1)^n n}{2n+1} \qquad (2)\, a_n = \frac{n^3}{3^n} \qquad (3)\, a_n = \frac{2^n + 1}{3^n + n^3} \qquad (4^*)\, a_n = \frac{n!}{n^n}$$

$$(5) a_{n+1} = \left(\frac{a_n}{2}\right)^2,\ a_0 < 4, \qquad (6) a_{n+1} = \left(\frac{a_n}{2}\right)^2,\ a_0 > 4.$$

$$(7^*) 4a_{n+1} = 5 - a_n^2,\ 0 < a_0 < 2, \qquad (8) a_{n+1} = \sqrt{2a_n - 1},\ a_0 > 1.$$

Fact needed for (4^*) **:** $\lim(1 + \frac{1}{n})^n = e \approx 2.71828.... > 1$.

Proof for (1). **Diverges.** We have

$$a_{2m} = \frac{2m}{4m+1} = \frac{1}{2 + \frac{1}{2m}}, \qquad a_{2m-1} = -\frac{2m-1}{4m-1} = -\frac{1 - \frac{1}{2m}}{2 - \frac{1}{2m}}.$$

Hence, Theorem 5.1 implies $\lim a_{2m} = \frac{1}{2}$ and $\lim a_{2m-1} = -\frac{1}{2}$. So, if we assume a_n converges to L, then we have $L = \frac{1}{2} = -\frac{1}{2}$ by the subsequence theorem and the uniqueness of the limit. Therefore, a_n can not converge. □

Proof for (2). **Converges to 0.** We have

$$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^3}{3n^3} = \frac{(1+\frac{1}{n})^3}{3}.$$

Hence, $\lim \frac{1}{n} = 0$ and Theorem 5.1 implies $\lim |\frac{a_{n+1}}{a_n}| = \frac{1}{3} < 1$. Therefore, the series $\sum a_n$ converges absolutely by the ratio test. Hence, $\lim a_n = 0$ by the test for divergence. □

Proof for (3). **Converges to 0.** We have

$$|\frac{a_{n+1}}{a_n}| = \frac{2^{n+1}+1}{2^n+1} \cdot \frac{3^n + n^3}{3^{n+1} + (n+1)^3} = \frac{2+\frac{1}{2^n}}{1+\frac{1}{2^n}} \cdot \frac{1+\frac{n^3}{3^n}}{3+3\frac{(n+1)^3}{3^{n+1}}}.$$

Theorem 3.4 shows $\lim \frac{1}{2^n} = 0$. Also, the proof for (2) shows $\lim \frac{n^3}{3^n} = 0$. Therefore, Theorem 5.1 implies $\lim |\frac{a_{n+1}}{a_n}| = \frac{2}{3} < 1$. Thus, the series $\sum a_n$ converges absolutely by the ratio test. Hence, $\lim a_n = 0$ by the test for divergence. □

Proof for (4). **Converges to 0.** We have $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$. Thus,

$$|\frac{a_{n+1}}{a_n}| = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n}.$$

Therefore, the fact $\lim(1+\frac{1}{n})^n = e$ and Theorem 5.1 implies $\lim |\frac{a_{n+1}}{a_n}| = e^{-1} < 1$. Thus, the series $\sum a_n$ converges absolutely by the ratio test. Hence, $\lim a_n = 0$ by the test for divergence. □

Proof for (5). **Converges to 0.** We have $a_1 = (a_0/2)^2 < (4/2)^2 = 4$ and $a_1 > 0$. Next, if $0 < a_k < 4$ holds for some integer k, then $a_{k+1} = (a_k/2)^2 < (4/2)^2 = 4$ and $a_{k+1} > 0$. Therefore, we have $0 < a_n < 4$ for all n by the mathematical induction.

Hence, $a_{n+1} = (a_n/2)^2 = a_n \cdot (a_n/4) < a_n$, namely a_n is decreasing. Therefore, a_n converges by the comparison theorem.

Let $L = \lim a_n$. Then, $L = \lim a_{n+1} = \lim a_n^2/4 = \frac{1}{4} \lim a_n^2 = \frac{1}{4}(\lim a_n)^2 = \frac{L^2}{4}$ by Theorem 5.1. Therefore, $L = 0$ or 4. However, we have $a_n < a_0$ and thus $L \leq a_0 < 4$ by the limit location theorem, namely $L \neq 4$. So, 0 is the limit. □

Proof for (6). **Diverges.** We have $a_1 = (a_0/2)^2 > (4/2)^2 = 4$. Next, if $a_k > 4$ holds for some integer k, then $a_{k+1} = (a_k/2)^2 > (4/2)^2 = 4$. Therefore, we have $a_n > 4$ for all n by the mathematical induction.

Hence, $a_{n+1} = (a_n/2)^2 = a_n \cdot (a_n/4) > a_n$, namely a_n is increasing.

We assume that a_n converges to L. Then, $L = \lim a_{n+1} = \lim a_n^2/4 = \frac{1}{4}\lim a_n^2 = \frac{1}{4}(\lim a_n)^2 = \frac{L^2}{4}$ by Theorem 5.1. Therefore, $L = 0$ or 4. However, $a_n > a_0$ and the limit location theorem shows $L \geq a_0 > 4$, namely $L \neq 0, 4$. Therefore, a_n diverges. $\square$

Proof for (7). **Converges to** 1. Let e_n denote $a_n - 1$. Then,

$$4e_{n+1} = 4a_{n+1} - 4 = 1 - a_n^2 = (1 - a_n)(1 + a_n) = -e_n(e_n + 2).$$

Since $|e_0| < 1$, we have $|e_1| = \frac{1}{4}|e_0||e_0+2| \leq \frac{1}{4}|e_0|(|e_0|+2) < \frac{3}{4}$. In addition, if $|e_k| < (\frac{3}{4})^k$ for some k, then

$$|e_{k+1}| = \frac{1}{4}|e_k||e_k + 2| \leq \frac{|e_k| + 2}{4}|e_k| < \frac{3}{4}\Big(\frac{3}{4}\Big)^k = \Big(\frac{3}{4}\Big)^{k+1}.$$

Thus, $-(\frac{3}{4})^n < e_n < (\frac{3}{4})^n$ holds for all n by mathematical induction. By Theorem 3.4, $\lim(\frac{3}{4})^n = 0$. Hence, by the squeeze theorem, $\lim e_n = 0$, namely $\lim a_n = 1$. $\square$

Proof for (8). **Converges to** 1. We have $a_1 = \sqrt{2a_0 - 1} > \sqrt{2 - 1} = 1$. Moreover, if $a_k > 1$ for some k, then $a_{k+l} = \sqrt{2a_k - 1} > \sqrt{2 - 1} = 1$. Therefore, by mathematical induction, $a_n > 1$ holds for all n.

Next, we observe

$$\begin{aligned} &a_{n+1} = \sqrt{2a_n - 1} \leq a_n \\ \Longleftrightarrow\ &2a_n - 1 \leq a_n^2 \qquad \text{(because } a_n, a_{n+1} > 0\text{)} \\ \Longleftrightarrow\ &0 \leq a_n^2 - 2a_n + 1 = (a_n - 1)^2. \end{aligned}$$

Therefore, a_n is decreasing. Hence, by the completeness property, a_n converges.

Let L be the limit of a_n. Then, $a_{n+1}^2 = 2a_n - 1$ and Theorem 5.1 shows

$$2L - 1 = \lim 2a_n - 1 = \lim a_{n+1}^2 = (\lim a_{n+1})^2 = L^2.$$

Thus, $(L - 1)^2 = L^2 - 2L + 1 = 0$, namely $L = 1$. $\square$

Problem 3. *Let $a_{n+1} = \frac{2}{1+a_n}$ and $a_0 > 1$.*

(1) *Show that the subsequence of even terms a_{2n} is decreasing and bounded below, and the subsequence of odd terms a_{2n-1} is increasing and bounded above.*
(2) *Show the convergence of a_n, and fine the limit.*

Proof for (1). We have

$$a_{n+2} = \frac{2}{1+a_{n+1}} = \frac{2}{1+\frac{2}{1+a_n}} = \frac{2(1+a_n)}{(1+a_n)+2} = \frac{2a_n+2}{a_n+3}.$$

Regarding a_{2n}, the condition $a_0 > 1$ yields

$$a_2 = \frac{2a_0+2}{a_0+3} > \frac{a_0+3}{a_0+3} = 1.$$

In addition, if $a_{2k} > 1$ for some k, then $a_{2k+2} = \frac{2a_{2k}+2}{a_{2k}+3} > \frac{a_{2k}+3}{a_{2k}+3} = 1$. So, by mathematical induction, $a_{2n} > 1$ holds for all n.

$$a_{2n+2} = \frac{2a_{2n}+2}{a_{2n}+3} \leq a_{2n} \iff 2a_{2n}+2 \leq a_{2n}^2 + 3a_{2n}$$
$$\iff 0 \leq a_{2n}^2 + a_{2n} - 2.$$

Thus, $a_{2n} > 1$ implies $a_{2n+2} \leq a_{2n}$. So, a_{2n} is decreasing and bounded below by 1.

Regarding a_{2n+1}, the condition $a_0 > 1$ yields $a_1 = \frac{2}{1+a_0} < \frac{2}{1+1} = 1$. So,

$$a_3 = \frac{2a_1+2}{a_1+3} < \frac{a_1+3}{a_1+3} = 1.$$

In addition, if $a_{2k+1} > 1$ for some k, then $a_{2k+3} = \frac{2a_{2k+1}+2}{a_{2k+1}+3} < \frac{a_{2k+1}+3}{a_{2k+1}+3} = 1$. So, by mathematical induction, $a_{2n+1} < 1$ holds for all n.

$$a_{2n+3} = \frac{2a_{2n+1}+2}{a_{2n+1}+3} \geq a_{2n+1} \iff 2a_{2n+1}+2 \geq a_{2n+1}^2 + 3a_{2n+1}$$
$$\iff 0 \geq a_{2n+1}^2 + a_{2n+1} - 2.$$

Thus, $a_{2n+1} < 1$ implies $a_{2n+3} \geq a_{2n+1}$. So, a_{2n+1} is increasing and bounded above.

□

Proof for (2). By the result of (1) and the completeness property, $\lim a_{2n} = L$ and $\lim a_{2n+1} = M$ exist. Moreover, the limit location theorem shows $L \geq 1$. Theorem 5.1 and $2 = a_{n+1}(1+a_n)$ lead to

$$2 = \lim a_{2n+1}(1+a_{2n}) = \lim a_{2n+1} + (\lim a_{2n+1})(\lim a_{2n}) = M + LM,$$

and

$$2 = \lim a_{2n}(1+a_{2n-1}) = \lim a_{2n} + (\lim a_{2n})(\lim a_{2n-1}) = L + LM.$$

So, $M = L = 2 - ML$, namely $\lim a_n = L$. Hence, $0 = L^2 + L - 2 = (L+2)(L-1)$, namely $L = 1$ or -2. Since $L \geq 1$, we have $\lim a_n = 1$.

□

Problem 4. *Let $a_{n+1} = \frac{1}{2+a_n}$ and $a_0 > 0$.*

(1) *Show that $\{a_n\}$ is a Cauchy sequence.*

(2) *Find the limit of* $\{a_n\}$ *and explain why it is the limit.*

Proof for (1). $a_0 > 0$ gives $a_1 = \frac{2}{1+a_0} > 0$. If $a_k > 0$ for some k, then $a_{k+1} = \frac{2}{1+a_k} > 0$. By mathematical induction, $a_n > 0$ for all n. Also,

$$|a_2 - a_1| = \Big|\frac{1}{2+a_1} - \frac{1}{2+a_0}\Big| = \frac{|a_1 - a_0|}{(2+a_0)(2+a_1)} < \frac{1}{4}|a_1 - a_0|.$$

If $|a_{k+1} - a_k| < (\frac{1}{4})^k |a_1 - a_0|$ for some k,

$$|a_{k+2} - a_{k+1}| = \Big|\frac{1}{2+a_{k+1}} - \frac{1}{2+a_k}\Big| = \frac{|a_{k+1} - a_k|}{(2+a_k)(2+a_{k+1})}$$
$$< \frac{1}{4}|a_{k+1} - a_k| < \Big(\frac{1}{4}\Big)^{k+1}|a_1 - a_0|.$$

By mathematical induction, $|a_{n+1} - a_n| < (\frac{1}{4})^n |a_1 - a_0|$ for all n. Hence, for $m > n$, we have

$$|a_m - a_n| \le \sum_{k=n}^{m-1} |a_{k+1} - a_k| \le \sum_{k=n}^{m-1} \Big(\frac{1}{4}\Big)^k |a_1 - a_0| \le \frac{|a_1 - a_0|}{4^n} \sum_{k=0}^{m-n-1} \Big(\frac{1}{4}\Big)^k$$
$$= \frac{|a_1 - a_0|}{4^n} \frac{1 - \frac{1}{4^{m-n-1}}}{1 - \frac{1}{4}} \le \frac{|a_1 - a_0|}{3} \Big(\frac{1}{4}\Big)^{n-1}.$$

By Theorem 3.4, given $\epsilon > 0$, $|a_m - a_n| < \epsilon$ holds for $n, m \gg 1$. □

Proof for (2). Since a_n is a Cauchy sequence, a_n converges to its limit L.

$$1 = \lim a_{n+1}(2 + a_n) = 2 \lim a_{n+1} + (\lim a_{n+1})(\lim a_n) = 2L + L^2,$$

namely $0 = L^2 + 2L - 1$. Hence, $L = -1 \pm \sqrt{2}$.

On the other hand, $a_n > 0$ implies $L \ge 0$. Thus, $L = -1 + \sqrt{2}$. □

Problem 5. *Let* S, T *be non-empty sets bounded above. Suppose* $s, t > 0$ *holds for all* $s \in S$ *and* $t \in T$*. Then, we have* $(\sup S)(\sup T) = \sup ST$*, where* $ST = \{st : s \in S, t \in T\}$*.*

Proof. By completeness property for sets, $\sup S$ and $\sup T$ exist. Since $s, t > 0$, $0 < s \le \sup S$ and $0 < t \le \sup T$ yields $st \le (\sup S)(\sup T)$, namely $(\sup S)(\sup T)$ is an upper bound for ST. Therefore,

$$\sup ST \le (\sup S)(\sup T).$$

For a fixed $s \in S$, $st \le \sup ST$ shows that $t \le (\sup ST)/s$ holds for all $t \in T$. Therefore, $\sup T \le (\sup ST)/s$, namely $s \sup T \le \sup ST$ holds for all $s \in S$. Since $\sup T \geq t > 0$, we have $s \leq (\sup ST)/(\sup T)$, namely $\sup S \leq (\sup ST)/(\sup T)$. Hence,

$$\sup ST \geq (\sup S)(\sup T).$$

In conclusion, $\sup ST = (\sup S)(\sup T)$.

$\square$

Problem 6. *Determine whether the following series are convergent or divergent, and explain why they are convergent or divergent.*

$$(1) \sum_{n=1}^{\infty} \frac{n}{2n+1} \qquad (2) \sum_{n=1}^{\infty} \frac{n^3}{3^n} \qquad (3) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \qquad (4) \sum_{n=1}^{\infty} \frac{2n}{n^2+1}.$$

Proof for (1). **Diverges.** $\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$, $\lim \frac{1}{n} = 0$, and Theorem 5.1 give

$$\lim \frac{n}{2n+1} = \lim \frac{1}{2+\frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2} \neq 0.$$

By the test for divergence, it diverges. $\square$

Proof for (2). **Converges.** We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\frac{(n+1)^3}{n^3} = \frac{1}{3}\lim\left(1+\frac{1}{n}\right)^3.$$

So, $\lim \frac{1}{n} = 0$ and Theorem 5.1 lead to $\lim |\frac{a_{n+1}}{a_n}| = \frac{1}{3} < 1$. Hence, by the ratio test, it converges. $\square$

Proof for (3). **Converges.** Since $\frac{1}{\sqrt{n+1}}$ is decreasing and converges to zero, the Cauchy's test guarantees the its alternating form is convergent. $\square$

Proof for (4). **Diverges.** The proof of Example 7.5A in the textbook shows $\sum \frac{1}{n}$ tends to infinity, namely $\sum \frac{2}{n}$ tends infinity. Moreover, we have

$$\lim \left|\frac{\frac{2}{n}}{\frac{2n}{n^2+1}}\right| = \lim \frac{n^2+1}{n^2} = \lim 1 + \frac{1}{n^2} = 1.$$

Therefore, by the asymptotic comparison test, $\sum \frac{2n}{n^2+1}$ also diverges. $\square$

Problem 7. *Find the radius of convergence of the following power series, and explain why.*

$$(1) \sum_{n=0}^{\infty} \frac{x^{2n}}{3^n} \qquad\qquad (2) \sum_{n=1}^{\infty} \frac{x^n}{2^n(n+1)}.$$

Proof for (1). For a fixed x, we have

$$\lim \sqrt[n]{|x^{2n}3^{-n}|} = \lim x^2/3 = x^2/3.$$

Hence, by the root test, the power series converges if $x^2/3 < 1$ and diverges if $x^2/3 > 1$, namely converges if $|x| < \sqrt{3}$ and diverges if $|x| > \sqrt{3}$. Hence, $\sqrt{3}$ is the radius of convergence. □

Proof for (2). If $x = 0$, it converges to zero. Given a fixed $x \neq 0$, we define $a_n = \frac{x^n}{2^n(n+1)}$. Then,

$$\Big|\frac{a_{n+1}}{a_n}\Big| = \frac{|x|^{n+1}}{|x|^n}\frac{2^n}{2^{n+1}}\frac{n+1}{n+2} = \frac{|x|}{2}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}.$$

Combining with $\lim \frac{1}{n} = 0$ and Theorem 5.1 yields $\lim |\frac{a_{n+1}}{a_n}| = \frac{|x|}{2}$. Hence, by the ratio test, the power series converges if $|x|/2 < 1$ and diverges if $|x|/2 > 1$, namely converges if $|x| < 2$ and diverges if $|x| > 2$. Hence, 2 is the radius of convergence. □

Problem 8 (Very challenging)**.** *Let $f(x)$ be a continuous function defined on $\mathbb{R}$. Suppose that $f(m2^{-n}) \geq 0$ holds for all integer $m \in \mathbb{Z}$ and natural number $n \in \mathbb{N}$. Show that $f(x) \geq 0$ holds for all $x \in \mathbb{R}$.*

Proof. Let $\mathbb{Y}$ denote the set of $m2^{-n}$. Given a point x, we will construct a sequence $y_n \in \mathbb{Y}$ converging to x. First of all, we let y_0, y_1 be the integers $m, m+1$ such that $m \leq x < m+1$, and let $I_1 = [y_0, y_1] = [a_1, b_1]$. Next, we set $y_2 = \frac{a_1+b_1}{2} = \frac{2m-1}{2} \in \mathbb{Y}$. Then, x must be contained in at least one of $[a_1, y_2]$ and $[y_2, b_1]$. So, we let $I_2 = [a_2, b_2]$ be the half interval containing x, and then $|b_2 - a_2| = \frac{1}{2}$.

We assume that we can iterate the halving process until obtaining $I_k = [a_k, b_k]$ with $a_k, b_k \in \mathbb{Y}$ and $|b_k - a_k| = \frac{1}{2^{k-1}}$. Then, we have $a_k = p_k 2^{-q_k}$ and $b_k = r_k 2^{-s_k}$ for some integers p, r_k and natural numbers q_k, s_k. Hence, we can set

$$y_{k+1} = \frac{a_k + b_k}{2} = \frac{p_k 2^{s_k} + r_k 2^{q_k}}{2^{q_k+s_k+1}} \in \mathbb{Y}.$$

Then, x must be contained in at least one of $[a_k, y_{k+1}]$ and $[y_{k+1}, b_k]$. So, we let $I_{k+1} = [a_k, b_k]$ be the half interval containing x, and then $|b_{k+1} - a_{k+1}| = \frac{1}{2^k}$ and $a_{k+1}, b_{k+1} \in \mathbb{Y}$.

Hence, by mathematical induction, there exist a nested sequence of closed intervals $I_n = [a_n, b_n]$ with $a_n, b_n \in \mathbb{Y}$ and $|b_n - a_n| = \frac{1}{2^{n-1}}$. Since Theorem 3.4 implies $\lim \frac{1}{2^{n-1}} = 0$, the nested interval theorem shows that there exists $L \in I_n$ and $\lim a_n = \lim b_n = L$. Moreover, $a_n \leq y_{n+1} \leq b_n$ and the squeeze theorem give $\lim y_n = L$. In addition, we have $x = L$, because $x, L \in I_n$

shows $|x - L| < \frac{1}{2^{n-1}}$ for all n. In conclusion, there exists $y_n \in \mathbb{Y}$ such that $\lim y_n = x$.

Since $f(x)$ is continuous, Theorem 11.5A shows $\lim f(y_n) = f(x)$. Hence, $f(y_n) \geq 0$ and the limit location theorem for sequences, we have the desired result $f(x) \geq 0$. □

Problem 9. *Let $f(x)$ be a bounded function defined on $\mathbb{R}$, and let $F(x) = \int_0^x f(t)dt$. Show that $F(x)$ is continuous on $\mathbb{R}$.*

Proof. Since $f(x)$ is bounded, $|f(x)| < M$ for some $M > 0$. Hence,

$$|F(x) - F(y)| = \Big| \int_0^x f(t)dt - \int_0^y f(t)dt \Big| = \Big| \int_y^x f(t)dt \Big|.$$

We set $f_+(t) = \max\{f(t), 0\} \geq 0$ and $f_-(t) = \max\{-f(t), 0\} \geq 0$. Then, $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Without loss of generality, we assume $y \geq x$. Then, the triangle inequality shows

$$\begin{aligned} \Big| \int_y^x f(t)dt \Big| &= \Big| \int_y^x f_+(t)dt - \int_y^x f_-(t)dt \Big| \leq \Big| \int_y^x f_+(t)dt \Big| + \Big| \int_y^x f_-(t)dt \Big| \\ &= \int_y^x f_+(t) + f_-(t)dt = \int_y^x |f(t)|dt < \int_y^x M dt = M|y - x|. \end{aligned}$$

Therefore, given $\epsilon > 0$, we have $|F(x_0) - F(x)| < M|x_0 - x| < \epsilon$ for $x \in (x_0 - \epsilon/M, x_0 + \epsilon/M)$. Namely, $F(x)$ is continuous at any point $x_0 \in \mathbb{R}$. □

Problem 10. *Suppose that $f(x)$ is a continuous function defined on $\mathbb{R}$, and $f(x) \geq 0$ holds for all $x \in \mathbb{R}$. Show that $g(x) = \sqrt{f(x)}$ is continuous on $\mathbb{R}$.*

Proof. If $f(x_0) = 0$ at some point x_0. Since $f(x)$ is continuous, given $\epsilon > 0$, $|f(x)| < \epsilon^2$ holds for $x \approx x_0$, namely $|g(x)| < \epsilon$ for $x \approx x_0$.

If $f(x_0) > 0$ at some point x_0. Then,

$$|g(x_0) - g(x)| = \Big| \frac{f(x_0) - f(x)}{\sqrt{f(x_0)} + \sqrt{f(x)}} \Big| \leq \frac{|f(x_0) - f(x)|}{\sqrt{f(x_0)}}.$$

Since $f(x)$ is continuous, given $\epsilon > 0$, $|g(x_0) - g(x)| < \epsilon\sqrt{f(x_0)}$ holds for $x \approx x_0$, namely $|g(x_0) - g(x)| < \epsilon$ for $x \approx x_0$. □

Problem 11. *Suppose that a continuous function $f(x)$ is defined on $[a, b]$ with $a \neq b$, and $f(x)$ is strictly increasing on (a, b). Show that $f(x)$ is strictly increasing on $[a, b]$.*

Proof. Assume that $f(b) \le f(c)$ for some $c \in (a,b)$. Since f is strictly increasing on (a,b), we have $f(b) \le f(c) < f(m)$ where $m = \frac{b+c}{2}$. On the other hand, $f(m) < f(x)$ for all $x \in (m,b)$. Therefore, the limit location theorem yields $f(b) = \lim_{x \to b^{-1}} f(x) \ge f(m)$, which contradicts to $f(m) > f(c) \ge f(b)$. Hence, $f(b) > f(c)$ for all (a,b), namely $f(x)$ is strictly increasing on $(a,b]$.

In the same manner, we can show that $f(x)$ is strictly increasing on $[a,b]$ by assuming that $f(a) \ge f(c)$ for some $c \in (a,b]$. □

Problem 12. *Suppose that $f(x), g(x)$ are continuous functions defined on $\mathbb{R}$. Show that the function $h(x) = \max\{f(x), g(x)\}$ is continuous on $\mathbb{R}$.*

Proof. Given a point x_0, we assume $f(x_0) \ge g(x_0)$ without loss of generality, namely $h(x_0) = f(x_0)$.

If $f(x_0) = g(x_0)$, then given $\epsilon > 0$ we have $|f(x) - h(x_0)| < \epsilon$, $|g(x) - h(x_0)| < \epsilon$ for $x \approx x_0$. Then, $|h(x) - h(x_0)| < \epsilon$ for $x \approx x_0$, because $h(x)$ is one of $f(x)$ and $g(x)$.

If $\sigma = f(x_0) - g(x_0) > 0$, then we have $|f(x) - f(x_0)| < \sigma/2$, $|g(x) - h(x_0)| < \sigma/2$ for $x \approx x_0$. Thus,

$$\begin{aligned} f(x) - g(x) &= \big(f(x_0) - g(x_0)\big) + \big(f(x) - f(x_0)\big) + \big(g(x_0) - g(x)\big) \\ &\ge \sigma - |f(x) - f(x_0)| - |g(x) - g(x_0)| > 0, \end{aligned}$$

holds for $x \approx x_0$, namely $f(x) = h(x)$ for $x \approx x_0$. Hence, $h(x)$ is continuous for $x \approx x_0$. □

Sample Exam

Solutions for problem 1-5 are given in the proofs above. Sure, the exam problems will be a bit different form the practice problems.

For problems 6 and 7, let me give you answer keys and hints. You can try to prove them during the spring break. The bonus problems in the exam will have totally different style from the practice problems and sample exam problems.

6. Possible Answers: $x^2 \sin(1/x)$, $\int_x^1 \sin(1/t)dt$.
7. Hint: consider the 2^{nd} order derivative of $1 + x + \cdots + x^n$ and plug $x = \frac{1}{2}$.